Cohomology of invertible sheaves on Projective spaces

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Let A be a noetherian ring. Recall $\mathbb{P}^d_A := \mathrm{Proj}(A[x_0 \ldots x_d])$ with the $\mathcal{O}(m) := A[x_0 \ldots x_d](m)^{\sim}$ are invertible sheaves on $\mathbb{P}^d_A$

Thm. Let $S = A[x_0 \ldots x_d]$ be the standard graded ring i.e $\deg a = 0$ if $a \in A$, $\deg(x_i) = 1$ $\forall i$. $X = \mathrm{Proj}(S)$

① The natural map $S_m \longrightarrow H^0(X, \mathcal{O}_X(m))$ is an isom $\forall m \geq 0$. For $m < 0$, $H^0(X, \mathcal{O}_X(m)) = 0$

② For $m > -d-1$, a) $H^d(X, \mathcal{O}_X(m)) = 0$

b) $H^d(X, \mathcal{O}_X(-d-1)) \xrightarrow{\sim} A$

c) There is a A-bilinear pairing for each $m \geq 0$

$$H^0(X, \mathcal{O}_X(m)) \times H^d(X, \mathcal{O}_X(-d-1-m)) \longrightarrow H^d(X, \mathcal{O}_X(-d-1)) \xrightarrow{\sim} A$$

which induces an isom

$$H^d(X, \mathcal{O}_X(-d-1-m)) \xrightarrow{\sim} \mathrm{Hom}_A(H^0(X, \mathcal{O}_X(m)), A)$$

③ For any $m \in \mathbb{Z}$, $H^j(X, \mathcal{O}_X(m)) = 0$ $\forall$ $0 < j < m$

④ If $j > d$, $H^j(X, \mathcal{F}) = 0$ for any q.coh sheaf $\mathcal{F}$ on X

Pf. ① was an exercise.

We compute the sheaf cohomologies of $\mathcal{O}_X(m)$ using the Čech complex given by cover

$$X = \bigcup_{i=0}^{d} D_+(x_i). \text{ Denote } \mathcal{U} = \{D_+(x_i)\}_{i=0,1,\ldots,d}$$

④ Since $C^j(\mathcal{U}, \mathcal{F}) = 0$ $\forall$ $j > d$, and

$$H^j(X, \mathcal{F}) = \check{H}^j(X, \mathcal{F}) \quad \forall j$$, we are done.

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Consider $\mathcal{G} = \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_X(m) \in \mathrm{QCoh}(X)$. $\mathcal{G}$ is a $\mathbb{Z}$-graded sheaf of $\mathcal{O}_X$-mods. So $C^\bullet(\mathcal{U},\mathcal{G})$ is a complex of graded $\mathcal{O}_X$-mods: $C^j(\mathcal{U},\mathcal{G})_m := C^j(\mathcal{U},\mathcal{O}_X(m))$, and the boundary maps of $C^\bullet(\mathcal{U},\mathcal{G})$ takes the m-th graded piece of $C^j(\mathcal{U},\mathcal{G})$ to the m-th graded piece of $C^{j+1}(\mathcal{U},\mathcal{F})$. Thus the cohomology gps of $\underline{C}^\bullet(\mathcal{U},\mathcal{G})$ are graded and

$$H^j(C^\bullet(\mathcal{U},\mathcal{G}))_m =: \check{H}^j(X,\mathcal{G})_m = \check{H}^j(\mathcal{U},\mathcal{O}_X(m)) \overset{\text{Thm}}{=} H^j(X,\mathcal{O}_X(m))$$

$$\Gamma(D_+(x_{i_0}\cdots x_{i_j}),\mathcal{G})$$

$$= \bigoplus_{m\in\mathbb{Z}} \Gamma(D_+(x_{i_0}\cdots x_{i_j}),\mathcal{O}_X(m))$$

$$= \bigoplus_{m\in\mathbb{Z}} \left(S(m)\left[\frac{1}{x_{i_0}\cdots x_{i_j}}\right]\right)_0$$

$$= \bigoplus_{m\in\mathbb{Z}} S\left[\frac{1}{x_{i_0}\cdots x_{i_j}}\right]_m = S\left[\frac{1}{x_{i_0}\cdots x_{i_j}}\right] \text{ as graded objects}$$

$$\begin{array}{ccccc} & C^{d-1}(\mathcal{U},\mathcal{G}) & & C^d(\mathcal{U},\mathcal{G}) & \\ & \| & & & \\ \xrightarrow{\quad} & \bigoplus_{j=0}^{d} S\left[\frac{1}{x_0\cdots\hat{x}_j\cdots x_d}\right] & \xrightarrow{\quad\quad} & S\left[\frac{1}{x_0\cdots x_d}\right] & \longrightarrow 0 \end{array}$$

The boundary map is given by alternating sum of the inclusion maps

$$S\left[\frac{1}{x_0\cdots\hat{x}_j\, x_d}\right] \longrightarrow S\left[\frac{1}{x_0\cdots x_n}\right]$$

So the coker of ∂ is isom as an graded S-mod to

$$\bigoplus_{\substack{a_i\in\mathbb{Z}_{>0}\\ \forall i}} A\cdot\left[\frac{1}{x_0^{a_0}\cdots x_d^{a_d}}\right],$$ $[\;]$ denotes the class in the quotient.

$\bigoplus_{\substack{a_i \in \mathbb{Z}_{>0} \\ \forall i}} A \cdot \left[x_0^{a_0} \cdots x_d^{a_d} \right]$, [ ] denotes the class in the quotient.

Then a) $\operatorname{coker}(*)_m = \tilde{H}^d(X, \mathcal{O}_X(m)) = 0$ if $m > -(d+1)$

b) if $m = -(d+1)$

$$H^d(X, \mathcal{O}_X(-d-1)) = \operatorname{coker}(*)_{-(d+1)} = A \cdot \left[\frac{1}{x_0 \cdots x_d} \right]$$

For $m \leq -(d+1)$, we get

$$H^d(X, \mathcal{O}_X(m)) \simeq \bigoplus_{\substack{a_0 + \cdots + a_d = m \\ a_i \in \mathbb{Z}_{>0} \ \forall}} A \left[\frac{1}{x_0^{a_0} \cdots x_d^{a_d}} \right]$$

c) The natural multiplication map

$$S^{\cdot} \times S\left[\frac{1}{x_0 \cdots x_d} \right] \longrightarrow S\left[\frac{1}{x_0 \cdots x_d} \right]$$

induces a map

$$\begin{array}{ccc} & H^d(X, \mathcal{F}) & H^d(X, \mathcal{F}) \\ & \| & \| \\ S \times & \operatorname{coker} * & \longrightarrow \operatorname{coker} * \end{array}$$

Restricting this we get an A-lin pairing

$$\begin{array}{ccccc} H^0(X, \mathcal{O}_X(m)) = S_m & \times & H^d(X, \mathcal{O}_X(-d-1-m)) & \longrightarrow & H^d(X, \mathcal{O}_X(-d-1)) \\ & & \| & & \wr\| \\ & & H^d(X, \mathcal{F})_{-d-1-m} & & A \cdot \left[\frac{1}{x_0 \cdots x_d} \right] \end{array}$$

Note that $H^d(X, \mathcal{F})_{-d-1-m} = H^d(X, \mathcal{O}_X(-d-1-m))$ is a free mod with basis of

$$\left\{ \frac{1}{x_0^{a_0} \cdots x_d^{a_d}} \;\middle|\; a_i \geq 1, \ \underset{= -(d+1) - m}{a_0 + \cdots + a_d} \right\}$$

$$= \left\{ \frac{1}{x_0 \cdots x_d} \cdot \frac{1}{x_0^{b_0} \cdots x_d^{b_d}} \;\middle|\; b_j \geq 0, \ \sum b_j = m \right\}$$

Thus the above pairing is perfect.

(3) We prove by inducing on d.

When $d = 1$, nothing to prove.

When $d=1$, nothing to prove.

For $d>1$, consider the exact seq

$$0 \to S(-1) \xrightarrow{x_0} S \to \frac{S}{x_0 S} \to 0$$

Applying $\sim$: $\mathrm{Mod}_S^{gr} \to \mathrm{Qcoh}(\mathcal{O}_X)$, get

$$0 \to \mathcal{O}_X(-1) \xrightarrow{x_0} \mathcal{O}_X \to i_*(\mathcal{O}_Y) \to 0 \qquad \text{——— } (\dagger)$$

where $Y = \mathrm{Proj}\left(\frac{A[x_0, \dots, x_d]}{x_0}\right) = \mathbb{P}_A^{d-1}$ and

$i : Y \hookrightarrow X$ is the closed immersion induced by

$A[x_0, \dots, x_d] \longrightarrow \frac{A[x_0 \dots x_d]}{x_0}$.

Take $(\dagger) \otimes \mathcal{O}_X(m)$ — since $\mathcal{O}_X(m)$ is loc free
$- \otimes \mathcal{O}_X(m)$ is exact. Get

$$0 \to \mathcal{O}_X(m-1) \xrightarrow[x_0]{} \mathcal{O}_X(m) \to i_*(\mathcal{O}_Y(m)) \to 0$$

The long exact seq of cohomology groups give

Take $\bigoplus_{m \in \mathbb{Z}}$ of the above exact sequences;

denote $\bigoplus_{m \in \mathbb{Z}} \mathcal{O}_Y(m)$ by $\mathcal{S}_Y$. This gives an exact seq

$$0 \to \mathcal{S}(-1) \to \mathcal{S} \to i_*(\mathcal{S}_Y) \to 0$$

Here $\mathcal{S}(-1)$ is the sheaf $\mathcal{S}$ whose grading is shifted by -1; i.e $[\mathcal{S}(-1)]_m = \mathcal{S}_{m-1}$ $\forall m \in \mathbb{Z}$

The above exact seq induces a long exact seq of cohomology groups.

$$\begin{array}{l} 0 \to H^0(X, \mathcal{S}(-1)) \xrightarrow{x_0} H^0(X, \mathcal{S}) \to H^0(Y, \mathcal{S}_Y) \\ \to H^1(X, \mathcal{S}(-1)) \xrightarrow{x_0} H^1(X, \mathcal{S}) \to H^1(Y, \mathcal{S}_Y) \end{array}$$

$$\to H^1(X, \mathcal{F}(-1)) \xrightarrow{x_0} H^1(X, \mathcal{F}) \to H^1(Y, \mathcal{F}_Y)$$

$$\to H^{j-1}(Y, \mathcal{F}_Y)$$

$$\to H^j(X, \mathcal{F}(-1)) \xrightarrow{x_0} H^j(X, \mathcal{F}) \to H^j(Y, \mathcal{F}_Y)$$

- For $m > 0$,

$$\begin{array}{ccccccccc} & & H^0(X,\mathcal{F}(-1))_m & & H^0(X,\mathcal{F})_m & & H^0(Y,\mathcal{F}_Y)_m & & \\ & & \| & & \| & & \| & & \\ 0 & \to & H^0(X, \mathcal{O}_X(m-1)) & \to & H^0(X, \mathcal{O}_X(m)) & \to & H^0(Y, \mathcal{O}_Y(m)) & \to & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ & & A[X_0, \dots X_d]_{m-1} & \xrightarrow{x_0} & A[X_0 \dots, X_d]_m & \to & A[X_1 \dots, X_d]_m & & \end{array}$$

is exact

- A similar argument shows that the seq below is exact $\forall m$.

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$$\begin{array}{ccccccccc} 0 & \to & H^{d-1}(Y, \mathcal{F}_Y)_m & \to & H^d(X, \mathcal{F}(-1))_m & \xrightarrow{x_0} & H^d(X, \mathcal{F})_m & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & H^{d-1}(Y, \mathcal{O}_Y(m)) & \to & H^d(X, \mathcal{O}(m-1)) & \xrightarrow{x_0} & H^d(X, \mathcal{O}_X(m)) & \to & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 0 & \leftarrow & H^0(Y, \mathcal{O}_Y(-d-m)) & \leftarrow & H^0(X, \mathcal{O}_X(-d-1-m+1)) & \xleftarrow{x_0} & H^0(X, \mathcal{O}_X(-d-1-m)) & \leftarrow & 0 \end{array}$$

Thus we conclude that

$$H^j(X, \mathcal{F}) = H^j(X, \mathcal{F}(-1)) \xrightarrow{x_0} H^j(X, \mathcal{F})$$

is an automorphism $\forall\ j \geq 1$.

We finish the proof using the following

<u>Claim</u>: The localization of the S-mod $H^j(X, \mathcal{F})$ at $\{1, x_0, x_0^2 \dots\}$ is zero $\forall\ j \geq 1$

i.e $\quad H^j(X, \mathcal{F})\left[\frac{1}{x_0}\right] = 0 \quad \forall j \geq 1$

<u>Pf</u> Recall that the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of S-mods.

$C^\bullet(\mathcal{U},\mathcal{G})$ is a complex of S-mods.

Note The complex

$(**)$ $0 \to C^0(\mathcal{U},\mathcal{G})[\frac{1}{x_0}] \to C^1(\mathcal{U},\mathcal{G})[\frac{1}{x_0}] \to \cdots \to C^d(\mathcal{U},\mathcal{G})[\frac{1}{x_0}] \to 0$

is the Čech complex of $\mathcal{O}_{\mathrm{Spec}(A[x_0,\dots,x_n][\frac{1}{x_0}])}$

with respect to the covering

$$\left\{ \mathrm{Spec}\left(A[x_0,\dots,x_n]\left[\tfrac{1}{x_i}\right]\right)\right\}_{i=0,\dots,d}$$

So $H^j(**) = 0$ $\forall j \geq 1$ — since q-coh sheaves on an affine scheme does not have any higher cohomologies.

But localization is an exact functor.

So $H^j(**) = H^j(C^\bullet(\mathcal{U},\mathcal{G}))[\frac{1}{x_0}] = 0$ $\forall j \geq 1$

Thm. Let A be a noetherian. $X \to \mathrm{Spec}(A)$ be projective, $\mathcal{L}$ be ample on X;

$\mathcal{F}$ be a coh $\mathcal{O}_X$-mod

(1) $H^i(X,\mathcal{F})$ are f.g A mod $\forall i$

(2) $\exists\, n_{\mathcal{F}}$ s.t $\forall\, n \geq n_{\mathcal{F}}$ and $i>0$, $H^i(X,\mathcal{F}\otimes\mathcal{L}^n) = 0$

for $n \geq n_{\mathcal{F}}$

Pf. Choose $m \in \mathbb{N}$ s.t $\mathcal{L}^m$ is very ample. So we have a closed immersion

$$\varphi : X \hookrightarrow \mathbb{P}^d_A \quad \text{s.t} \quad \varphi^*(\mathcal{O}_{\mathbb{P}^d_A}(1)) \simeq \mathcal{L}^m.$$

(1) Since $R^i\varphi_* = 0$ $\forall i>0$, for any $\mathcal{G} \in \mathrm{Qcoh}(X)$,

$H^i(X,\mathcal{G}) \simeq H^i(\mathbb{P}^d, \varphi_*\mathcal{G})$ $\forall i$.

Moreover since $\varphi_*(\mathrm{Coh}(X)) \subseteq \mathrm{Coh}(\mathbb{P}^d_A)$, w.l.o.G we assume in (1), that $X = \mathbb{P}^d_A$ and $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^d_A)$

Moreover since $\varphi_*(\mathrm{Coh}(X)) \subseteq \mathrm{Coh}(\mathbb{P}^d_A)$, w.l.o.g. we assume in (1), that $X = \mathbb{P}^d_A$ and $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^d_A)$

Set $\mathcal{O}(j) = \mathcal{O}_{\mathbb{P}^d}(j)$

<u>Prop</u>: Every coherent sheaf $\mathcal{G}$ on $\mathbb{P}^d_A$ can be written as a quotient of a <u>finite</u> direct sum

$$\bigoplus \mathcal{O}(j)$$ [The same $\mathcal{O}(j)$ may appear multiple times].

<u>Pf</u>. For $0 \le i \le d$, suppose $m_{i1} \dots m_{in_i} \in \Gamma(D_+(x_i), \mathcal{G})$ generate $\Gamma(D_+(x_i), \mathcal{G})$ as $\mathcal{O}_{\mathbb{P}^d}(D_+(x_i))$ mod.

Realizing $x_i \in \Gamma(\mathbb{P}^d_A, \mathcal{O}(1))$, recall that $\exists\ n_0$ s.t

$m_{ij} \otimes x_i^{n_0}$ extends to a global section $\tilde{m}_{ij} \in \Gamma(\mathbb{P}^d_A, \mathcal{G}(n_0))$

for $1 \le i \le d$, $1 \le j \le n_i$

Consider
$$\bigoplus^{n_0 + \dots + n_d} \mathcal{O}_{\mathbb{P}^d_A} \longrightarrow \mathcal{G}(n_0)$$
$$1 \longmapsto \tilde{m}_{ij}$$

This induces $\bigoplus^{n_0 + \dots + n_d} \mathcal{O}_{\mathbb{P}^d_A}(-n_0) \longrightarrow \mathcal{G}$ which is sur.

Indeed on $D_+(x_i)$
$$m_{ij} = (m_{ij} \otimes x_i^{n_0}) \otimes \frac{1}{x_i^{n_0}}$$
$$= \tilde{m}_{ij} \otimes \frac{1}{x_i^{n_0}} \quad \text{and} \quad \frac{1}{x_i^{n_0}} \in \mathcal{O}_{\mathbb{P}^d}(-n_0).$$

is in the image $\forall\ 1 \le j \le n_i$. $\square$

Returning to (1), chose a surjection as in Prop

$$\bigoplus \mathcal{O}(j) = \mathcal{E} \longrightarrow \mathcal{F} \to 0$$

Suppose the kernel is $\mathcal{R}$, have an exact seq

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0.$$

We induce on $d+1-i$, to conclude (1):

When $d+1-i = 0$, since $H^{d+1}(\mathbb{P}^d_A, \mathcal{G}) = 0$, for any $\mathcal{G} \in \mathrm{Coh}(\mathbb{P}^d_A)$. Our induction hypothesis is that for

When $d+1-i=0$, since $H^{d+1}(\mathbb{P}^d_A, \mathcal{G})=0$, for any $\mathcal{G} \in \mathrm{Coh}(\mathbb{P}^d_A)$. Our induction hypothesis is that for $i \geq n+1$, and any $\mathcal{G} \in \mathrm{Coh}(\mathbb{P}^d)$,

$$H^i(\mathbb{P}^d, \mathcal{G}) \text{ is f.g } /A)$$

Now consider the long exact seq of cohomology groups coming from the above exact seq

$$H^n(\mathbb{P}^d, \mathcal{R}) \to H^n(\mathbb{P}^d, \mathcal{E}) \to H^n(\mathbb{P}^d, \mathcal{F}) \to H^{n+1}(\mathbb{P}^d, \mathcal{R})$$

$H^n(\mathbb{P}^d, \mathcal{E})$ f.g $/A$ by our explicit computation and $H^{n+1}(\mathbb{P}^d, \mathcal{R})$ f.g $/A$ by our induction hypothesis. So we are done. $\square$

(2) We use the same notation as in (1).

We first prove that for any $\mathcal{G} \in \mathrm{Coh}(\mathbb{P}^d_A)$:

$\exists\, n_{\mathcal{G}}$ s.t $H^i(\mathbb{P}^d, \mathcal{G}(n)) = 0 \quad \forall\, n \geq n_{\mathcal{G}}$ and $i>0$:

For that choose an exact seq

$$0 \to \mathcal{R}' \to \mathcal{E}' = \bigoplus \mathcal{O}(j) \to \mathcal{G} \to 0$$ as above and induce on $d+1-i$.

In the induction step, use the long exact seq coming from the short exact seq

$$0 \to \mathcal{R}'(l) \to \mathcal{E}'(l) \to \mathcal{G}(l) \to 0,$$

which gives

$$H^i(\mathbb{P}^d, \mathcal{E}'(l)) \to H^i(\mathbb{P}^d, \mathcal{G}(l)) \to H^{i+1}(\mathbb{P}^d, \mathcal{R}'(l)).$$

Since $\mathcal{E}'$ is a finite direct sum of $\mathcal{O}(j)$'s and for $\forall\, t>0$, $H^i(\mathbb{P}^d, \mathcal{O}(t)) = 0 \quad \forall\, i$, we get the conclusion for $X=\mathbb{P}^d$ and $\mathcal{L}=\mathcal{O}(1)$.

The conclusion for $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(1)$.

Deducing (2) from the case on $X = \mathbb{P}^d$, $\mathcal{L} = \mathcal{O}(1)$:

Projection formula: $f : (T, \mathcal{O}_T) \to (T', \mathcal{O}_{T'})$ maps of ringed spaces, $\mathcal{M} \in \mathrm{Mod}_{\mathcal{O}_T}$, $\mathcal{E}$ be a locally free $\mathcal{O}_{T'}$ mod.

Then there is a natural isom

$$f_* \mathcal{M} \otimes_{\mathcal{O}_{T'}} \mathcal{E} \xrightarrow{\sim} f_*(\mathcal{M} \otimes f^* \mathcal{E})$$

Pf. HW

Coming back to (2), applying projection formula along

$$\varphi : X \hookrightarrow \mathbb{P}^d_A, \quad \text{get} \quad \varphi_* \mathcal{F} \otimes \mathcal{O}(n) \xrightarrow{\sim} \varphi_*(\mathcal{F} \otimes \varphi^* \mathcal{O}(n))$$
$$\simeq \varphi_*(\mathcal{F} \otimes \mathcal{L}^{mn})$$

Again since
$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{mn}) \underset{P.F}{\simeq} H^i(\mathbb{P}^d, \varphi_*(\mathcal{F} \otimes \mathcal{L}^{mn})) \simeq H^i(\mathbb{P}^d, \varphi_* \mathcal{F} \otimes \mathcal{O}(n))$$

and $\varphi_* \mathcal{F}$ is coherent, we prove (2) when $\mathcal{L}$ is replaced by $\mathcal{L}^m$ and for any coh $\mathcal{O}_X$-mod $\mathcal{F}$.

Now applying this case to each of $\{\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \dots, \mathcal{F} \otimes \mathcal{L}^{m-1}\}$

get $n_{\mathcal{F}}$ s.t. $\forall\, n \geq n_{\mathcal{F}}$

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^j \otimes \mathcal{L}^{mn}) = 0 \quad \forall\, n \geq n_{\mathcal{F}} \;\; \forall\, i > 0$$
$$\forall\, j = 0, \dots, m-1$$

Claim: $\forall\, n \geq m n_{\mathcal{F}}$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \;\; \forall\, i > 0$.

Pf For $n \geq m n_{\mathcal{F}}$, we can write

$$n = mq + r \quad \text{where} \quad q \geq n_{\mathcal{F}} \text{ and } 0 \leq r \leq m-1.$$

So $\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^r) \otimes \mathcal{L}^{mq}$ where $0 \leq r \leq m-1$, $q \geq n_{\mathcal{F}}$.

So we are done by the line above the claim.

I I P 18.12.26 lecture

$\forall n \geq n_{\mathcal{F}}$.

So we are done by the line above the claim.

End of 18.12.24 lecture

Start of 20.12.24 lecture

The converse of the part (2) of the Thm above is true and provides a characterization of ampleness.

Thm. Let $X \to \operatorname{Spec}(A)$ be a proper map, $\mathcal{L}$ be an invertible sheaf. T.F.A.E

① $\mathcal{L}$ is ample

② For every $\mathcal{F} \in \operatorname{Coh}(X)$, $\exists\, n_{\mathcal{F}}$ such that $\forall n \geq n_{\mathcal{F}}$,

$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \forall\, n \geq n_{\mathcal{F}}, \ \forall\, i \geq 1$

③ For every $\mathcal{F} \in \operatorname{Coh}(X)$, $\exists\, n_{\mathcal{F}}$ such that $\forall n \geq n_{\mathcal{F}}$

$H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \forall\, n \geq n_{\mathcal{F}}$.

Pf (1) $\Rightarrow$ (2) follows from the Thm above, indeed any proper scheme over $\operatorname{Spec}(A)$ with an ample invertible sheaf is automatically projective.

(2) $\Rightarrow$ (3) clear

(3) $\Rightarrow$ (1)

Prop: Every quasi-compact scheme has a closed point

Pf. See notes.

Cor: Let Y be a quasi-compact scheme. Let $\{U_i\}_{i \in I}$ be a collection of opens such that $\cup U_i$ contains all the closed pts of Y; Then $Y = \cup U_i$.

Pf. $Z = Y - \bigcup_{i \in I} U_i$ is closed in Y, so as a top space is quasi-compact. Endow Z with a scheme structure. Z must have a closed pt. But a closed pt of Z

$\bigcup_{i \in I}$ ... is quasi-compact. Endow Z with a scheme structure. Z must have a closed pt. But a closed pt of Z is a closed point of Y not in $\bigcup_{i \notin I} U_i$.

For (3) $\Rightarrow$ (1), The next claim is crucial.

<u>Claim</u>. Let $\mathcal{F} \in \mathrm{Coh}(X)$. For a closed point $x \in X$, $\exists$ an affine open nbhd U_x of x and $n_x \in \mathbb{N}$ s.t $\forall n \geq n_x$

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow \Gamma(U_x, \mathcal{F} \otimes \mathcal{L}^n)$$

is surjective.

<u>Pf.</u> Consider x with the structure sheaf $k(x) = \frac{\mathcal{O}_{X,x}}{m_x}$.

<u>Then</u> $i : (x, k(x)) \longrightarrow (X, \mathcal{O}_X)$ is a closed immersion

Have an exact seq $0 \to \mathcal{I}_x \to \mathcal{O}_X \to i_x(k(x)) \to 0$.

This gives an exact seq

$(*)$ $\quad 0 \to \mathcal{I}_x \mathcal{F} \to \mathcal{F} \to i_x(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)) \to 0$

Note $\mathcal{I}_x \mathcal{F} \in \mathrm{Coh}(X)$.

Choose n_0 s.t $\forall n \geq n_0$,

$$H^1(X, \mathcal{I}_x \mathcal{O}_X \otimes \mathcal{L}^n) = H^1(X, \mathcal{I}_x \mathcal{F} \otimes \mathcal{L}^n) = 0$$

$(*) \Rightarrow \forall n \geq n_0$

$$H^0(X, \mathcal{L}^n) \to H^0(\{x\}, \mathcal{L}^n \otimes k(x))$$
$$= \mathcal{L}^n_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

and $$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(\{x\}, \mathcal{F} \otimes \mathcal{L}^n \otimes k(x))$$
$$= \mathcal{F}_x \otimes \mathcal{L}^n_x \otimes \frac{\mathcal{O}_{X,x}}{m_x}$$

are surjective.

<u>Recall</u>: Let R be a noeth ring, $N \to M$ maps of R-mods where M is f.g / R. For a prime $\mathfrak{p} \subseteq R$.

<u>Recall:</u> Let R be a noeth ring, $N \to M$ map of R-mods where M is f.g/R. For a prime $p \subseteq R$, if $\frac{N_p}{pN_p} \to \frac{M_p}{pM_p}$ is surj then

① $N_p \to M_p$ is surj, ② $\exists$ $f \in R$, $f \notin p$ s.t $N_f \to M_f$ is surj.

So, we can choose an affine open nbhds of x: $V, U_0, \ldots, U_{n_0-1}$

$$H^0(X, \mathcal{L}^{n_0}) \to \Gamma(V, \mathcal{L}^{n_0})$$

&

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^{n_0+j}) \to \Gamma(U_j, \mathcal{F} \otimes \mathcal{L}^{n_0+j})$$ are

surj $\forall$ $j = 0, \ldots, n_0 - 1$

So $H^0(X, \mathcal{L}^{n_0}) \otimes_A \mathcal{O}_{X|V} \longrightarrow \mathcal{L}^{n_0}_{|V}$ and

$H^0(X, \mathcal{F} \otimes \mathcal{L}^{n_0+j}) \otimes \mathcal{O}_{X|U_j} \to \mathcal{F} \otimes \mathcal{L}^{n_0+j}_{|U_j}$ are surj $\forall$ $j = 0, \ldots, n_0 - 1$

<u>Claim:</u> Set $U_x = V \cap U_0 \cap \cdots \cap U_{n_0-1}$,

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^j) \otimes_A \mathcal{O}_{X|U_x} \to \Gamma(U_x, \mathcal{F} \otimes \mathcal{L}^j)$$ is surj $\forall$ $j \geq n_0$

<u>Pf:</u>

Follows from the fact that: for two opens $V_1, V_2 \subseteq X$ and two sheaves $\mathcal{F}_1, \mathcal{F}_2$ if

$H^0(X, \mathcal{F}_1) \otimes_A \mathcal{O}_{V_1} \longrightarrow \mathcal{F}_{1|V_1}$ is surj

$H^0(X, \mathcal{F}_2) \otimes_A \mathcal{O}_{V_2} \longrightarrow \mathcal{F}_{2|V_2}$ is surj

Then $H^0(X, \mathcal{F}_1 \otimes \mathcal{F}_2) \otimes_A \mathcal{O}_{V_1 \cap V_2} \longrightarrow \mathcal{F}_1 \otimes \mathcal{F}_{2|V_1 \cap V_2}$ is surj.

Then $H^0(X, \mathcal{F}_1 \otimes \mathcal{F}_2) \otimes_A \mathcal{O}_{V_1 \cap V_2} \to \mathcal{F}_1 \otimes \mathcal{F}_2|_{V_1 \cap V_2}$ is sur.

Granting this fact, to prove the claim note that

for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^n \simeq \mathcal{F} \otimes \mathcal{L}^j \otimes \mathcal{L}^{n_0 q}$
where q and j are the unique integers s.t

$$n = n_0 q + j \quad \text{and} \quad 0 \leq j \leq n_0 - 1.$$

This proves the claim.

For each closed pt $x \in X$, produce a pair (U_x, n_0) as above. Since $\bigcup_{x \in X} U_x$ contains all the closed points of X, $X = \cup U_x$. But X is quasi-compact.

So $X = \bigcup_{i=1}^{m} U_{x_i}$. Let n' be the max of the n_0's coming from the x_i's.

Then $\forall\, n \geq n'$,

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \otimes_A \mathcal{O}_X|_{U_{x_i}} \longrightarrow \mathcal{F} \otimes \mathcal{L}^n|_{U_{x_i}}$$

is sur $\forall\, i = 1, \ldots, m$

As $X = \bigcup_{i=1}^{m} U_{x_i}$, $H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \otimes \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{L}^n$ is sur $\Rightarrow$ $\mathcal{F} \otimes \mathcal{L}^n$ is globally gen $\forall\, n \geq n_0$.